How a Superior Elementary Math Textbook Teaches Division by Fractions

Mathematics for Elementary Teachers with Activities
Beckmann, 3rd ed., pp. 250-254

6.4 Fraction Division from the "How Many Groups?" Perspective

F·P **Focal Points**
Grade 6

Have you ever wondered why we "invert and multiply" to divide fractions? In this section, we will see why this standard procedure for dividing fractions gives answers to fraction division problems that agree with what we expect from the meaning of division. We will also study story problems that can be solved with fraction division.

We'll start by recalling the "invert and multiply" or "multiply by the reciprocal" procedure for fraction division. Then we'll examine what fraction division means from the "how many groups?" perspective. Finally, we'll consider several related ways why the fraction division procedure is valid.

You might wonder why we need several different explanations for why a mathematical fact is true. One reason is that teachers should know multiple explanations in order to reach more students effectively. But explanations are also important in mathematics not only because they prove that mathematical facts really are true, but because they show how ideas are connected and related to each other. In mathematics, the paths through which ideas are linked and interconnected are interesting in their own right.

The "Invert and Multiply" or "Multiply by the Reciprocal" Procedure

The procedure for dividing fractions is straightforward. To divide fractions, such as

$$\frac{3}{4} \div \frac{2}{3} \text{ and } 6 \div \frac{2}{5}$$

we can use the familiar "invert and multiply" method in which we invert the divisor and multiply by it, as in

$$\frac{3}{4} \div \frac{2}{3} = \frac{3}{4} \cdot \frac{3}{2} = \frac{3 \cdot 3}{4 \cdot 2} = \frac{9}{8} = 1\frac{1}{8}$$

and

$$6 \div \frac{2}{5} = \frac{6}{1} \div \frac{2}{5} = \frac{6}{1} \cdot \frac{5}{2} = \frac{6 \cdot 5}{1 \cdot 2} = \frac{30}{2} = 15$$

reciprocal

Another way to describe this "invert and multiply" method for dividing fractions is in terms of the reciprocal of the divisor. The **reciprocal** of a fraction $\frac{C}{D}$ is the fraction $\frac{D}{C}$. In order to divide fractions, we multiply by the reciprocal of the divisor. So, in general,

$$\frac{A}{B} \div \frac{C}{D} = \frac{A}{B} \cdot \frac{D}{C} = \frac{A \cdot D}{B \cdot C}$$

As an interesting special case, notice that we can apply the fraction division procedure to whole number division because every whole number is equal to a fraction (e.g., $2 = \frac{2}{1}$ and $3 = \frac{3}{1}$). Therefore,

$$2 \div 3 = \frac{2}{1} \div \frac{3}{1} = \frac{2}{1} \cdot \frac{1}{3} = \frac{2 \cdot 1}{1 \cdot 3} = \frac{2}{3}$$

Notice that this result, $2 \div 3 = \frac{2}{3}$, agrees with our finding earlier in this chapter on the link between division and fractions, namely, that $A \div B = \frac{A}{B}$.

The "How Many Groups?" Interpretation

Class Activity ***Now Turn to Class Activities Manual***

6R "How Many Groups?" Fraction Division Problems, p. 129

With the "how many groups?" interpretation of division, $8 \div 2$ means the number of groups we can make when we divide 8 objects into groups with 2 objects in each group. In other words, $8 \div 2$ tells us how many groups of 2 we can make from 8.

Similarly, with the "how many groups?" interpretation of division,

$$\frac{8}{3} \div \frac{2}{3}$$

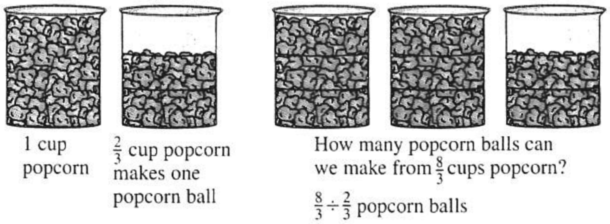
tells us how many groups of $\frac{2}{3}$ we can make from $\frac{8}{3}$. For example, suppose you are making popcorn balls and each popcorn ball requires $\frac{2}{3}$ cup of popcorn. If you have $2\frac{1}{3} = \frac{8}{3}$ cup of popcorn, then how many popcorn balls can you make? In this case you want to divide $\frac{8}{3}$ cup of popcorn into groups (balls) so that there is $\frac{2}{3}$ cup of popcorn in each group, as indicated in Figure 6.13. According to the "how many groups?" interpretation of division, you can make

$$\frac{8}{3} \div \frac{2}{3}$$

popcorn balls.

FIGURE 6.13

A "how many groups?" story problem for $\frac{8}{3} \div \frac{2}{3}$



Another way to analyze "how many groups?" fraction division is to reformulate fraction division problems as equivalent multiplication problems. From the "how many groups?" viewpoint,

$$8 \div 2 = ? \text{ corresponds to } ? \times 2 = 8$$

since we are asking how many groups of 2 are in 8. Therefore, similarly,

$$\frac{8}{3} \div \frac{2}{3} = ? \text{ corresponds to } ? \times \frac{2}{3} = \frac{8}{3}$$

and we are asking how many $\frac{2}{3}$s are in $\frac{8}{3}$.

Class Activity ***Now Turn to Class Activities Manual***

6S Dividing Fractions by Dividing the Numerators and Dividing the Denominators, p. 131

Using the "How Many Groups?" Interpretation to Explain Why "Invert and Multiply" Is Valid

There are many ways to explain why the "invert and multiply" procedure for dividing fractions is valid. One way, which we study next, involves reasoning about "how many groups?" fraction division problems. A side benefit of this way of reasoning is that it actually develops *another* valid procedure for dividing fractions. As we'll see, this other method consists of first giving the two fractions a common denominator and then dividing the numerators.

Consider the division problem

$$\frac{2}{3} \div \frac{1}{2}$$

The following is a story problem for this division problem:

> How many $\frac{1}{2}$ cups of water are in $\frac{2}{3}$ cup of water?

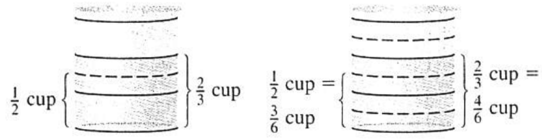
Or, said another way,

> How many times will we need to fill a $\frac{1}{2}$ cup measuring cup with water and pour it into a container that holds $\frac{2}{3}$ cup of water in order to fill the container?

From the diagram in Figure 6.14 we can say right away that the answer to this problem is "one and a little more" because one-half cup clearly fits in two-thirds of a cup, but then a little more is still needed to fill the two-thirds of a cup. But what is this "little more"? Remember the original question: How many $\frac{1}{2}$ cups of water are in $\frac{2}{3}$ cup of water? The answer should be of the form "so and so many $\frac{1}{2}$ cups of water." This means that we need to express this "little more" as *a fraction of* $\frac{1}{2}$ *cup of water*. How can we do that? By subdividing both the $\frac{1}{2}$ and the $\frac{2}{3}$ into common parts—namely, by using common denominators.

FIGURE 6.14

$\frac{2}{3} \div \frac{1}{2} = ?$ How many $\frac{1}{2}$ cups of water are in $\frac{2}{3}$ cup?



When we give $\frac{1}{2}$ and $\frac{2}{3}$ the common denominator of 6, then, as on the right of Figure 6.14, and as in Figure 6.15, the $\frac{1}{2}$ cup of water is made out of 3 parts (3 sixths cup of water), and the $\frac{2}{3}$ cup of water is made out of 4 parts (4 sixths cup of water), so the "little more" we were discussing in the previous paragraph is just one of those parts. Since $\frac{1}{2}$ cup is 3 parts, and the "little more" is 1 part, the "little more" is $\frac{1}{3}$ of the $\frac{1}{2}$ cup of water. This explains why $\frac{2}{3} \div \frac{1}{2} = 1\frac{1}{3}$: There's an entire $\frac{1}{2}$ cup plus another $\frac{1}{3}$ of the $\frac{1}{2}$ cup in $\frac{2}{3}$ cup of water.

To summarize, we are considering the fraction division problem $\frac{2}{3} \div \frac{1}{2}$ in terms of the story problem "how many $\frac{1}{2}$ cups of water are in $\frac{2}{3}$ cup of water?" If we give $\frac{1}{2}$ and $\frac{2}{3}$ the common denominator of 6, then we can rephrase the problem as "how many $\frac{3}{6}$ cup are in $\frac{4}{6}$ cup?" But in terms of Figures 6.14 and 6.15, this is equivalent to the problem "how many 3s are in 4?" which is the problem $4 \div 3 = ?$, whose answer is $\frac{4}{3} = 1\frac{1}{3}$.

The approach that we just developed actually provides us with another general procedure for dividing fractions! The reasoning we used was general and can be applied in exactly the same way to other fractions. Notice the process for dividing fractions that we developed: Starting with a fraction division problem, we first gave the fractions a common denominator; then we just divided the numerators of these new fractions, disregarding the common denominators. The same line of reasoning will work for any fraction division problem,

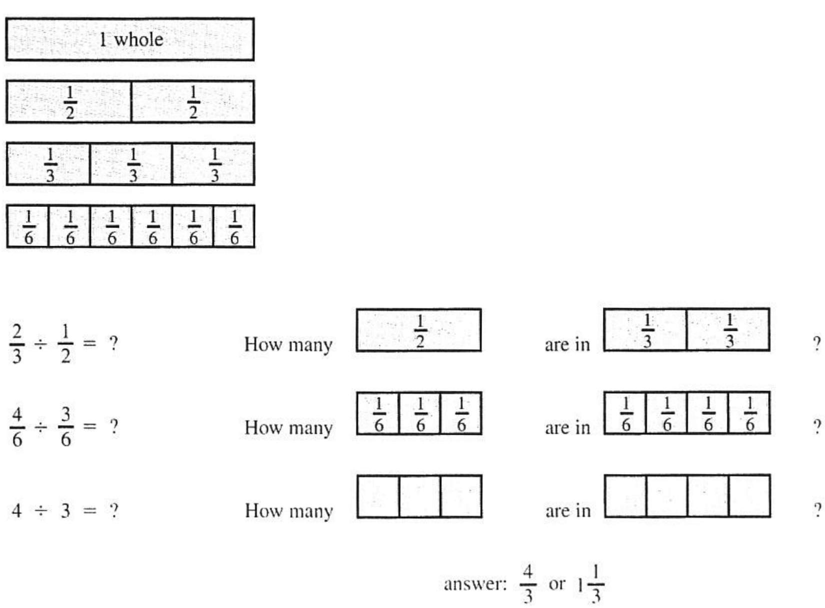
$$\frac{A}{B} \div \frac{C}{D}$$

Thinking logically, as before, and interpreting $\frac{A}{B} \div \frac{C}{D}$ as "how many $\frac{C}{D}$ cups of water are in $\frac{A}{B}$ cups of water?", we can conclude that

$$\frac{A}{B} \div \frac{C}{D} = \frac{A \cdot D}{B \cdot D} \div \frac{B \cdot C}{B \cdot D} = (A \cdot D) \div (B \cdot C) = \frac{A \cdot D}{B \cdot C}$$

FIGURE 6.15

$\frac{2}{3} \div \frac{1}{2} = ?$ How many $\frac{1}{2}$ are in $\frac{2}{3}$?



The final expression, $\frac{A \cdot D}{B \cdot C}$, is the answer provided by the "invert and multiply" procedure for dividing fractions. Therefore, we know that the "invert and multiply" procedure gives answers to division problems that agree with what we expect from the meaning of division.

Viewing Division Problems as Unknown Factor Multiplication Problems to Explain Why "Multiply by the Reciprocal" Is Valid

Another way to explain why the usual fraction division procedure is valid is to use the important link between division and multiplication. Recall that every division problem is equivalent to a multiplication problem with an unknown factor (actually two multiplication problems). Thus,

$$S \div T = ?$$

is equivalent to

$$? \div T = S$$

(or $T \cdot ? = S$). So, for example,

$$1 \div \frac{C}{D} = ? \quad \text{is equivalent to} \quad ? \cdot \frac{C}{D} = 1$$

What can we put in place of the ? to make either of these equations true? Notice that the reciprocal of $\frac{C}{D}$, namely $\frac{D}{C}$, will make the second equation true:

$$\frac{D}{C} \cdot \frac{C}{D} = \frac{D \cdot C}{C \cdot D} = 1 \qquad (6.7)$$

Equation (6.7) shows that when you multiply a fraction by its reciprocal, you get 1. Another way to say this is that the reciprocal of a fraction is the **multiplicative inverse** of the fraction.

In general, it is the fact that the reciprocal of a fraction is its multiplicative inverse that makes the "multiply by the reciprocal" procedure for fraction division work. Let's see why that is so. The equation

$$\frac{A}{B} \div \frac{C}{D} = ? \quad \text{is equivalent to} \quad ? \cdot \frac{C}{D} = \frac{A}{B}$$

What can we put in place of the ? to make either of these equations true? If we put

$$\frac{A}{B} \cdot \frac{D}{C}$$

in for the ? in the second equation, the $\frac{D}{C}$ and $\frac{C}{D}$ will multiply together to make 1, thus leaving us with $\frac{A}{B}$, which makes the equation true:

$$\left(\frac{A}{B} \cdot \frac{D}{C}\right) \cdot \frac{C}{D} = \frac{A}{B} \cdot \left(\frac{D}{C} \cdot \frac{C}{D}\right) = \frac{A}{B}$$

Therefore,

$$\frac{A}{B} \cdot \frac{D}{C}$$

really is the solution to the division problem

$$\frac{A}{B} \div \frac{C}{D} = ?$$

and we have developed another line of reasoning for why the "multiplying by the reciprocal" procedure for fraction division is valid.